MUHANDISLIK & IQTISODIYOT

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ijtimoiy-iqtisodiy, innovatsion texnik, fan va ta'limga oid ilmiy-amaliy jurnal

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05.00.00 - Texnika fanlari
08.00.00 - Iqtisodiyot fanlar

Elektron nashr,
454 sahifa, sentyabr, 2025-yil.

muhandislik & iqtisodiyot

ijtimoiy-iqtisodiy, innovatsion texnik,
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05.01.00 – Axborot texnologiyalari, boshqaruv va kompyuter grafikasi
05.01.01 – Muhandislik geometriyasi va kompyuter grafikasi. Audio va video texnologiyalari
05.01.02 – Tizimli tahlil, boshqaruv va axborotni qayta ishlash
05.01.03 – Informatikaning nazariy asoslari
05.01.04 – Hisoblash mashinalari, majmualari va kompyuter tarmoqlarining matematik va dasturiy ta'minoti
05.01.05 – Axborotlarni himoyalash usullari va tizimlari. Axborot xavfsizligi
05.01.06 – Hisoblash texnikasi va boshqaruv tizimlarining elementlari va qurilmalari
05.01.07 – Matematik modellashtirish
05.01.11 – Raqamli texnologiyalar va sun'iy intellekt
05.02.00 – Mashinasozlik va mashinashunoslik
05.02.08 – Yer usti majmualari va uchish apparatlari
05.03.02 – Metrologiya va metrologiya ta'minoti
05.04.01 – Telekommunikasiya va kompyuter tizimlari, telekommunikasiya tarmoqlari va qurilmalari. Axborotlarni taqsimlash
05.05.03 – Yorug'lik texnikasi. Maxsus yoritish texnologiyasi
05.05.05 – Issiqlik texnikasining nazariy asoslari
05.05.06 – Qayta tiklanadigan energiya turlari asosidagi energiya qurilmalari
05.06.01 – To'qimachilik va yengil sanoat ishlab chiqarishlari materialshunosligi
05.08.03 – Temir yo'l transportini ishlatish
05.09.01 – Qurilish konstruksiyalari, bino va inshootlar
05.09.04 – Suv ta'minoti. Kanalizatsiya. Suv havzalarini muhofazalovchi qurilish tizimlari
10.00.06 – Qiyosiy adabiyotshunoslik, chog'ishtirma tilshunoslik va tarjimashunoslik
10.00.04 – Yevropa, Amerika va Avstraliya xalqlari tili va adabiyoti
08.00.01 – Iqtisodiyot nazariyasi
08.00.02 – Makroiqtisodiyot
08.00.03 – Sanoat iqtisodiyoti
08.00.04 – Qishloq xo'jaligi iqtisodiyoti
08.00.05 – Xizmat ko'rsatish tarmoqlari iqtisodiyoti
08.00.06 – Ekonometrika va statistika
08.00.07 – Moliya, pul muomalasi va kredit
08.00.08 – Buxgalteriya hisobi, iqtisodiy tahlil va audit
08.00.09 – Jahon iqtisodiyoti
08.00.10 – Demografiya. Mehnat iqtisodiyoti
08.00.11 – Marketing
08.00.12 – Mintaqaviy iqtisodiyot
08.00.13 – Menejment
08.00.14 – Iqtisodiyotda axborot tizimlari va texnologiyalari
08.00.15 – Tadbirkorlik va kichik biznes iqtisodiyoti
08.00.16 – Raqamli iqtisodiyot va xalqaro raqamli integratsiya
08.00.17 – Turizm va mehmonxona faoliyati

Ma'lumot uchun, OAK Rayosatining 2024-yil 28-avgustdagi 360/5-son qarori bilan "Dissertatsiyalar asosiy ilmiy natijalarini chop etishga tavsiya etilgan milliy ilmiy nashrlar ro'yxati"ga texnika va iqtisodiyot fanlari bo'yicha "Muhandislik va iqtisodiyot" jurnali ro'yxatga kiritilgan.

Muassis: "Tadbirkor va ishbilarmon" MChJ

Hamkorlarimiz:

1. Toshkent shahridagi G.V.Plexanov nomidagi Rossiya iqtisodiyot universiteti
2. Toshkent davlat iqtisodiyot universiteti
3. Toshkent irrigatsiya va qishloq xo'jaligini mexanizatsiyalash muhandislari instituti" milliy tadqiqot universiteti
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NONPARAMETRIC SEQUENTIAL CONFIDENCE ESTIMATION BY FIXED-WIDTH CONFIDENCE INTERVALS: ASYMPTOTIC CONSISTENCY AND ASYMPTOTIC EFFICIENCY

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Abstract: This article investigates nonparametric sequential estimation of functionals of an unknown distribution using fixed-width confidence intervals. General conditions for the asymptotic consistency of the intervals and the asymptotic efficiency of stopping times are established. These conditions are verified through sequential interval estimation of the unknown probability density of a linear random process. The approach is based on limit theorems for randomly stopped stochastic processes, providing rigorous asymptotic guarantees.

Keywords: random variable, stopping time, confidence interval, fixed width, asymptotic consistency, asymptotic efficiency, probability density, linear random process.

Annotatsiya: Ushbu maqolada notaraqsimli izchil statistik baholashning fiksirlangan kenglikdagi ishonch intervallari orqali amalga oshirilishi o‘rganilgan. Noma’lum taqsimot funksiyasi funksionallarini izchil baholash uchun asimptotik moslik va to‘xtash momentining asimptotik samaradorligiga oid umumiy shartlar keltirilgan. Ushbu shartlar chiziqli tasodifiy jarayonning ehtimollik zichligini izchil interval baholash misolida tekshirilgan. Natijada tasodifiy to‘xtatilgan stoxastik jarayonlarga oid chekli teoremalarga asoslangan yondashuvning afzalliklari asoslab berilgan.

Kalit so‘zlar: tasodifiy kattalik, to‘xtash momenti, ishonch intervali, fiksirlangan kenglik, asimptotik moslik, asimptotik samaradorlik, ehtimollik zichligi, chiziqli tasodifiy jarayon.

Аннотация: В статье рассматривается последовательная оценка функционалов неизвестной функции распределения с помощью доверительных интервалов фиксированной ширины. Получены общие условия асимптотической согласованности таких интервалов и асимптотической эффективности момента остановки. Эти условия проверены на примере последовательной интервальной оценки неизвестной плотности распределения линейного случайного процесса. Подход основан на предельных теоремах для случайно остановленных стохастических процессов, что обеспечивает строгие асимптотические результаты.

Ключевые слова: случайная величина, момент остановки, доверительный интервал, фиксированная ширина, асимптотическая согласованность, асимптотическая эффективность, плотность распределения, линейный случайный процесс.

INTRODUCTION

Sequential statistical methods have become essential in modern data-driven environments where observations arrive gradually and the stopping time of analysis must adapt to the behavior of the data. Unlike classical fixed-sample procedures, sequential confidence estimation allows researchers to determine the sample size dynamically, ensuring that inference is guided by precision requirements rather than arbitrary pre-specified observation counts. Within this framework, constructing fixed-width confidence intervals for nonparametric targets presents unique theoretical challenges, particularly due to the absence of distributional assumptions and the dependence of stopping rules on observed sample paths.

Recent advances highlight the need for procedures that guarantee both asymptotic consistency of the resulting estimators and asymptotic efficiency of the stopping rules. These properties ensure that the confidence intervals converge to the true parameter with the desired coverage probability and that the expected stopping time approaches the theoretical lower bound. This paper investigates nonparametric sequential confidence estimation under fixed-width constraints, develops conditions ensuring the strong asymptotic properties of the procedures, and contributes to the growing literature on adaptive and distribution-free inference. By establishing rigorous theoretical guarantees, the study provides a foundation for designing practical sequential algorithms suitable for real-time systems, large-scale monitoring, and data streams where flexibility and statistical robustness are equally critical.

REVIEW OF LITERATURE ON THE SUBJECT

J.Chow and N.Robbins [2] developed the basic asymptotic theory of sequential confidence estimation by fixed-width intervals in the general case. They considered the problem of estimating the average unknown distribution under the sole assumption of the existence of a second distribution moment. The problem of confidence estimation of the average value by fixed-width intervals for a distribution that is attracted to a stable distribution that does not have a second-order moment is considered in the author's work [3].

A comprehensive review of works on sequential nonparametric estimation by fixed-width intervals published before 1997 is given in the monograph by M.Ghosh, N. Makhopadhyay and P. Sen [4].

Of the subsequent works, it is worth noting the work of T. Shiohama and S.Tanagushi [5], where the point and interval sequential estimation of the integral functional from an unknown spectral density of a Gaussian stationary process is studied. The work by N. Makhopadhyay and D.Silva [6] provides an overview of applications of sequential estimation methods, including nonparametric estimation by fixed-width intervals.

In scientific papers published in recent years [7-13], asymptotic problems of sequential statistical analysis have been studied, in [9-13] scientific articles - problems of sequential interval estimation, and in scientific articles [7,8] - problems of sequential point estimation.

Scientific articles [13-16] have investigated the problems of sequential point estimation of exponential, gamma, and Bernoulli distributions. Consistent nonparametric point estimation of the hazard function has been studied in scientific articles [17-19].

Since [2], many authors have used the classical theorem F. Anscombe [21] in specific statistical models to prove the asymptotic validity of confidence intervals of a fixed width. on the marginal distribution of random sequences with random indexes, which assumed the fulfillment of the condition of "uniform continuity in probability". An overview of such works is given in the monograph [4]. An alternative approach to proving the asymptotic consistency of fixed-width confidence intervals based on the weak principle of invariance is considered in detail in P.Sen [22].

In our work, to prove the asymptotic consistency of confidence intervals of a fixed width, we used an approach based on the application of limit theorems for randomly stopped stochastic processes developed by D.Silvestrov [23]. The first part of the paper presents general conditions for the asymptotic consistency of a fixed-width confidence interval and the asymptotic efficiency of the stopping moment in sequential interval estimation of a wide class of functionals $\theta(F)$ from an unknown distribution function $F(x), x \in R_1$ from a family Ψ of distribution functions satisfying certain regularity conditions. In the second part of the paper, these general conditions are verified by sequential interval estimation of an unknown probability density of a linear random process.

RESEARCH METHODOLOGY

The research methodology is based on data obtained from sequential observations of independent random variables and their functional estimators within the chosen statistical models. The analytical process relies on sequential analysis techniques, where asymptotic properties of estimators, random stopping rules, and limit theorems are applied to assess the consistency and efficiency of fixed-width confidence intervals.

ANALYSIS AND RESULTS

Unlike classical methods of mathematical statistics, in which the number of observations is fixed in advance, sequential analysis methods are characterized by the fact that the moment of termination of observations (the moment of stopping) is random and is determined depending on the values of the observed data and on the adopted measure of optimality of the constructed statistical estimate in the statistical model under study.

For a simple statistical model, we present the basic asymptotic problem of sequential estimation by fixed-width intervals.

On the probability space (Ω,F,P) consider a sequence of independent equally normally distributed random variables $\xi_1,\xi_2,\dots,\xi_n,\dots$ with mean θ and variance σ^2. Using the empirical mean $\theta_n=\frac{1}{n}\sum_{i=1}^{n}\xi_i$, we will construct an interval estimate $I(n)=[\theta_n-\varepsilon,\theta_n+\varepsilon]$ for the unknown mean θ, here ε > 0 is a small number.

Forthesignificancelevel $0<\gamma<1$,wefindthequantile $\grave{a}_\gamma=\hat{O}^{-1}\left(\frac{1+\gamma}{2}\right)$, here $\hat{O}(\tilde{o})=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\tilde{o}}\mathring{a}^{-\frac{\delta^2}{2}}dy$.Then

$$P\{\theta\in I(n_\varepsilon)\}=P\left\{\left|\frac{\sum_{i=1}^{n_\varepsilon}(\xi_i-\theta)}{\sigma\sqrt{n_\varepsilon}}\right|\le\frac{\varepsilon\sqrt{n_\varepsilon}}{a\sigma}\cdot a\right\}\ge P\left\{\frac{1}{\sqrt{n_\varepsilon}}\left|\sum_{i=1}^{n_\varepsilon}\frac{\xi_i-\theta}{\sigma}\right|\le a\right\}=2\;(\;)-1=\gamma$$

the inequality

$$n_\varepsilon=\inf\left(n\ge1:n\ge\frac{a^2\sigma^2}{\varepsilon^2}\right)$$

holds for an integer .

Therefore, $I(n_\varepsilon)$ it is an interval estimate for an unknown mean with a confidence level of $0<\gamma<1$.

If the variance σ^2 of the normal distribution is unknown, then, as shown in the work of G.Danzig [1], it is impossible to construct a fixed-width confidence interval for the mean θ based on a sample of a fixed non-random volume.

In this case, the task of consistent estimation by confidence intervals of a fixed width is reduced to the task of choosing a rule or stopping moment

$$N_\varepsilon=\inf\left(n\ge1:n\ge\frac{a^2}{\varepsilon^2}\sigma_n^2\right)$$

, where σ_n^2 a consistent estimate for an unknown variance σ^2 and finding conditions for fulfillment $\lim\limits_{\varepsilon\to0}P\{\theta\in I(N_\varepsilon)\}\ge\gamma$ (asymptotic consistency of the confidence interval $I(N_\varepsilon)$). Many researchers have been engaged in solving this problem.

Specific statistical models in which the approach of this work based on the application of limit theorems for randomly stopped stochastic processes is used to prove the asymptotic validity of confidence intervals of a fixed width are considered in [24-26].

1. Sequential interval estimation of a functional from an unknown distribution function.

Let us consider random variables $\xi_1,\xi_2,\dots,\xi_n$ with an unknown distribution function $F(x),x\in R_1$ from on the probability space (Ω, F, P), where Ψ is a family of distribution functions satisfying certain regularity conditions. To evaluate the functional $\theta(F)$ of the distribution function $F(\;)$, we consider a consistent estimate $\theta_n(F)=\theta_n(\xi_1,\xi_2,\dots,\xi_n)$ that satisfies the conditions:

(A): the decomposition is valid for the estimate $\theta_n(F)$:

$$\theta_n(F)=\theta(F)+[\theta_n(F)-E(\theta_n(F))]+[E(\theta_n(F))-\theta(F)]=\theta(F)+\sum_{k=1}^{n}Y_n(F,\xi_k)+Z_n$$

,(1) here the random variables $Y_n(F,\xi_k),1\le k\le n$ and the bias Z_n are such that there are numbers $\alpha>0$ and $\sigma^2(F)>0$, which means as $n\to\infty$

$$n^\alpha\sum_{k=1}^{n}Y_n(F,\xi_k)\xrightarrow{D}N(0,\sigma^2(F))\quad \text{и}\quad n^\alpha Z_n\longrightarrow0,$$

where $N(0,\sigma^2(F))$ is a normally distributed random variable with mean 0, variance $\sigma^2(F)$, and the symbol $\xrightarrow{D}$ means weak convergence of random variables.

For the significance level $0<\gamma<1$ we find the quantile $\grave{a}_\gamma = \hat{O}^{-1}\left(\frac{1+\gamma}{2}\right)$ and for a small number ε > 0, we denote the integer

$$n_\varepsilon = \inf\left(n \geq 1 : n \geq \left(\frac{a^2\sigma^2(F)}{\varepsilon^2}\right)^{\frac{1}{2\alpha}}\right) \tag{2}$$

and construct the 2ε length $I(n_\varepsilon)=[\theta_{n_\varepsilon}(F)-\varepsilon, \theta_{n_\varepsilon}(F)+\varepsilon]$ interval . Then, from condition (A) and from Slutsky's theorem [27], we obtain $\sigma^{-1}(F)n^\alpha(\theta_n(F)-\theta(F)) \xrightarrow{D} N(0,1)$ as $n\to\infty$. Therefore,

$$P\{\theta(F)\in I(n_\varepsilon)\} = P\left\{\left|\theta_{n_\varepsilon}(F)-\theta(F)\right| \leq \varepsilon\right\} = P\left\{\sigma^{-1}(F)n_\varepsilon^\alpha\left|\theta_{n_\varepsilon}(F)-\theta(F)\right| \leq \frac{\sigma^{-1}(F)n_\varepsilon^\alpha\varepsilon}{a}a\right\} \geq$$

$$\geq \hat{\boldsymbol{\Theta}}\left\{\dot{\boldsymbol{\sigma}}^{-1}(F)n_\varepsilon^\alpha\left|\theta_{n_\varepsilon}(F)-\theta(F)\right| \leq a\right\} \to 2\quad(\quad)-1=\gamma$$ as $\varepsilon\to 0$.

This sequential procedure has the disadvantage that n_ε it depends on unknown functionality $\sigma^2(F)$, so it is necessary to evaluate it. Let 's assume a $V_n^2 = V_n^2(\xi_1,\xi_2,...,\xi_2)$ reasonable estimate for the unknown $\sigma^2(F)$. Based on (2), we define the moment of stopping N_ε as follows

$$N_\varepsilon = \inf\left(n \geq 1 : n \geq \left(\frac{a^2V_n^2}{\varepsilon^2}\right)^{\frac{1}{2\alpha}}\right),$$

ε > 0. (3)

Then the sequential confidence interval for the unknown , based on the moment of stopping (3), has the form $I(N_\varepsilon)=[\theta_{N_\varepsilon}(F)-\varepsilon, \theta_{N_\varepsilon}(F)+\varepsilon],$ ε > 0. Following X. Chow and N. Robbins [2] introduce the following asymptotic criteria:

Definition 1.1. A fixed-width confidence interval $I(N_\varepsilon)$ is called asymptotically consistent if $\lim_{\varepsilon\to 0} P\{\theta(F)\in I(N_\varepsilon)\} \geq \gamma$ holds for a confidence level of $0<\gamma<1$ and for all $F(x)\in\Psi$.

Definition 1.2. The stopping moment N_ε is called asymptotically effective if $I(N_\varepsilon)$ is asymptotically consistent and $\lim_{\varepsilon\to 0}\frac{\mathring{A}(N_\varepsilon)}{n_\varepsilon}=1.$

Note that the number n_ε is equal to the minimum number of observations necessary for the asymptotic probability as $\varepsilon\to 0$ covering the interval $I(n_\varepsilon)=[\theta_{n_\varepsilon}(F)-\varepsilon, \theta_{n_\varepsilon}(F)+\varepsilon]$ of the "known" functional to be at least the confidence level $0<\gamma<1$. Therefore, the asymptotic efficiency in the sense of the Chow and Robbins stopping moment N_ε means that the average stopping moment N_ε for small ε behaves "the same" as in the case of "well - known" functionality $\theta(F)$.

Theorem 1.1. If $V_n^2 \to \sigma^2(F)$ with probability 1 as $n\to\infty$, then:

1) For any $\varepsilon>0$ $P\{N_\varepsilon<\infty\}=1$;

2) $N_\varepsilon\to\infty$ with a probability of 1 as $\varepsilon\to 0$;

3) $\lim_{\varepsilon\to 0} E(N_\varepsilon)=\infty$

4) $\frac{N_\varepsilon}{n_\varepsilon}\to 1$ with a probability of 1 as $\varepsilon\to 0$.

$$P\{N_\varepsilon=\infty\}=\lim_{n\to\infty}P\{N_\varepsilon\ge n\}\le\lim_{n\to\infty}P\left\{n\le\left(\frac{a^2V_n^2}{\varepsilon^2}\right)^{\frac{1}{2\alpha}}\right\}=0.$$

Proof. 1) For any ε > 0

2) It can be seen from (3) that N_ε increases with decreasing ε and $N_\varepsilon\to\infty$ with probability 1 as $\varepsilon\to 0$.

3) From the monotone convergence theorem we get $\lim\limits_{\varepsilon\to 0}E(N_\varepsilon)=\infty$.

4) From (2) and (3), there are inequalities for any ε > 0

$$\frac{(\varepsilon^2)^{\frac{1}{2\alpha}}}{(\varepsilon^2)^{\frac{1}{2\alpha}}+\left(a^2\sigma^2(F)\right)^{\frac{1}{2\alpha}}}\le\frac{1}{n_\varepsilon}\le\left(\frac{\varepsilon^2}{a^2\sigma^2(F)}\right)^{\frac{1}{2\alpha}}$$ and $$\left(\frac{a^2V_{N_\varepsilon}^2}{\varepsilon^2}\right)^{\frac{1}{2\alpha}}\le N_\varepsilon\le 1+\left(\frac{a^2V_{N_\varepsilon-1}^2}{\varepsilon^2}\right)^{\frac{1}{2\alpha}}.$$

Since $N_\varepsilon\to\infty$ as $\varepsilon\to 0$ and $V_n^2\to\sigma^2(F)$ with probability 1 as $\varepsilon\to 0$, then

$$\left(\frac{V_{N_\varepsilon}^2}{\sigma^2(F)}\right)^{\frac{1}{2\alpha}}\to 1$$ and $$\left(\frac{V_{N_\varepsilon-1}^2}{\sigma^2(F)}\right)^{\frac{1}{2\alpha}}\to 1$$ with probability 1 as $\varepsilon\to 0$. Hence, the last statement of the theorem is obtained from the relation

$$\frac{\left(V_{N_\varepsilon}^2\right)^{\frac{1}{2\alpha}}}{\left(\varepsilon^2a^{-2}\right)^{\frac{1}{2\alpha}}+\left(\sigma^2(F)\right)^{\frac{1}{2\alpha}}}\le\frac{N_\varepsilon}{n_\varepsilon}\le\frac{1}{n_\varepsilon}+\left(\frac{V_{N_\varepsilon-1}^2}{\sigma^2(F)}\right)^{\frac{1}{2\alpha}}.$$

Theorem 1.2. If for some number $\varepsilon_0>0$ the series $\sum\limits_{m=1}^{\infty}\sup\limits_{0\le\varepsilon\le\varepsilon_0}P\left\{\frac{N_\varepsilon}{n_\varepsilon}\ge m\right\}$ converges, then $\lim\limits_{\varepsilon\to 0}\frac{E\left(N_\varepsilon\right)}{n_\varepsilon}=1.$

To prove the statement of Theorem 1.2, it is sufficient to show (by virtue of Theorem 2.1 [28]) the uniform integrability of the set, but as shown in [29] (Lemma 3.2), it is sufficient to fulfill the condition of Theorem 1.2.

Theorem 1.3. Let the condition of Theorem 1.1 be fulfilled. and

1) $n^\alpha\left(E(\theta_n(F))-\theta(F)\right)\to 0$ as $n\to\infty$;

2) $\zeta_\varepsilon(t),t\in[0,1]\xrightarrow{J}W(t),t\in[0,1]$ as $\varepsilon\to 0$, here, the symbol $\xrightarrow{J}$ means weak convergence in the Skorokhod J-topology, $\zeta_\varepsilon(t)=\sigma^{-1}(F)\cdot n_\varepsilon^\alpha\cdot\sum\limits_{k=1}^{[n_\varepsilon t]}Y_{n_\varepsilon}(F,\xi_k),t\in[0,1]$ and $W(t),t\in[0,1]$ the Wiener process with $E(W(t))=0,E(W^2(t))=1$. Then:

1) $\sigma^{-1}(F)N_\varepsilon^\alpha\left(\theta_{N_\varepsilon}(F)-\theta(F)\right)\xrightarrow{D}N(0,1)$ as $\varepsilon\to 0$;

2) $\lim\limits_{\varepsilon\to 0}P\left(\theta\left(F\right)\in I(N_\varepsilon)\right)\ge\gamma$, that is, the confidence interval of a fixed width $I(N_\varepsilon)$ is asymptotically consistent.

To prove the statements of Theorem 1.3., we note that from 2) the conditions of Theorem 1.3. and 4) the statements of Theorem 1.1. we obtain that for a random process $\zeta_\varepsilon(t),t\in[0,1]$ and for a random variable $\frac{N_\varepsilon}{n_\varepsilon}$, the conditions of Theorem 2.8.1 [23] on the limiting distribution of randomly stopped stochastic processes are fulfilled. From this theorem, 1) the conditions of Theorem 1.3 and from decomposition (1) we obtain the statements of Theorem 1.3.

Remark 1.1. If the conditions of Theorems 1.2 and 1.3 are fulfilled, then the stopping moment (3) is asymptotically efficient in the sense of Chow and Robbins.

2. Sequential interval estimation of the probability density of a linear random process.

Let us consider a strictly stationary linear random process on the probability space (Ω,F,P), represented as follows

$$X_t = \sum_{n=0}^{n} g_n Y_{t-n}, \qquad (4)$$

where $Y_t, t \in Z = \{\dots,-1,0,1,\dots\}$ is a sequence of independent identically distributed random variables and $\{g_n, n \ge 0\}$ a sequence of numbers such that $g_n \to 0$ for $n \to \infty$. Random processes of the form (4) describe, for example, autoregressive and mixed moving average autoregressive schemes.

Denote by $f\left(x_1, x_2, \dots, x_k\right)$ the joint probability density of a random vector $\left(X_1, \dots, X_k\right)$. Assume that the marginal probability density $f(x), x \in (-\infty,\infty)$ is unknown. To estimate the probability density $f(x)$ at a point x, consider the kernel type estimate

$$f_n(x) = \frac{1}{nh(n)} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h(n)}\right) \qquad (5)$$

where $\hat{E}(x), x \in (-\infty,\infty)$ is a non-negative and bounded function, $\{h(n), n \ge 1\}$ a sequence of real numbers such that $h(n) \to 0$ for $n \to \infty$.

Remark 2.1. The estimate (5) depends on the decreasing sequence of numbers $\{h(n), n \ge 1\}$ chosen by the statistician. The optimal sequence $\{h(n), n \ge 1\}$, that is, one in which the root-mean-square error has the smallest order of magnitude, depends on an unknown probability density and its derivatives. Methods for choosing the optimal sequence $\{h(n), n \ge 1\}$ based on observations and balancing the spread and bias of the estimate are given, for example, in the monograph by E. Nadaray [30].

The estimate (5) allows decomposition

$$\tilde{Q}_n(x) \ne f(x) + \sum_{i=1}^{n} Y_n(K, \ {}_i) + \ {}_n,$$

где $\tilde{O}_n(K, \ {}_i) = K\frac{1}{nh(n)}\left(\frac{x-X_i}{h(n)}\right) - \left(\hat{E}\frac{1}{nh(n)} K\left(\frac{x-X_i}{h(n)}\right)\right), \ h \le \ \le$ и $Z_n = E\left(f_n(x)\right) - f(x)$.

We denote $k(u) = \int_{-\infty}^{\infty} \exp(iuy)K(y)dy, k_q = \lim_{u\to 0}\frac{1-k(u)}{|u|^q}$ and let $\varphi(\grave{e})$ and $\psi(\grave{e})$ be characteristic functions of random variables Y_1 and X_1. Let's introduce the conditions

(A): $\int_{-\infty}^{\infty} K(y)dy < \infty$, $\hat{E} \ne du \int_{-\infty}^{\infty} K^2(\ ) < \infty$; $\lim_{|y|\to\infty} |y| K(y) = 0, \int_{-\infty}^{\infty} |K(y+h) - K(y)|dy < c_1 h$, where c_1 is the final positive constant;

(B): $\int_{-\infty}^{\infty} |u\varphi(u)|du < \infty$, $k_q < \infty$, $\left|\int_{-\infty}^{\infty} \exp(-iux)|u|^q \psi(u)du\right| < \infty$ for some $q > 0$;

(C): $E(|Y_1|^m) < \infty$ for some $m > 0$ and if $m \ge 1$, then $E(Y_1) = 0$;

(D): $\sum_{i=n}^{\infty} i|g_i|^{\alpha} = O(n^{-\gamma})$ for some $\gamma > 0$, where $\alpha = \frac{m}{2}$, if $m \le 1$ and $\alpha = \frac{1}{2}$, if $m > 1$.

(H): $\tilde{h}(n) = \ ^{-\beta}, 0 < C\beta < 1, \ > 0$

Remark 2.2. The constant C in condition (H) is chosen based on observations. For example, E.Parzen

[31] showed that for $\beta_0 = \dfrac{1}{2q+1}$ a constant C has the form $\tilde{N} = \tilde{N}(f(x), f^{(q)}(x)) = \dfrac{kf(x)}{(\sqrt{2q} \mid k_q f^{(q)}(x))^{\beta_0}}$ and it is not difficult to prove that if $f_n(x)$ and $f_n^{(q)}(x)$ are consistent estimates for $f(x)$ and $f^{(q)}(x)$, respectively, then $\tilde{N}_n = \tilde{N}(f_n(x), f_n^{(q)}(x)) = \dfrac{kf_n(x)}{(\sqrt{2q} \mid k_q f_n^{(q)}(x))^{\beta_0}}$ is a consistent estimate for C.

In this case $\beta_0 < \beta < 1$ the problem of choosing a constant C based on observations remains open, at least within the scope of this work.

Theorem 2.1. [K.Chanda 32] Let conditions (A), (B), (C), (D), (H) and $\dfrac{1}{2q+1} < \beta < 1$ be fulfilled. Then $\sigma^{-1} n^{\frac{1-\beta}{2}} \sum_{i=1}^{n} Y_n(K, X_i) \xrightarrow{D} N(0,1)$ and $n^{\frac{1-\beta}{2}} Z_n \to 0$ for $n \to \infty$, where . $\sigma^2 = k \cdot f(x).$

For the significance level $0 < \gamma < 1$ we find the quantile $\grave{a}_\gamma = \hat{O}^{-1}\left(\dfrac{1+\gamma}{2}\right)$ and for any number ε > 0 we construct

$$n_\varepsilon = \inf\left(n \ge 1 : n \ge \left(\frac{a^2 \sigma^2}{\varepsilon^2}\right)^{\frac{1}{1-\beta}} \right). \qquad (6)$$

For the 2ε length interval $I(n_\varepsilon) = [f_{n_\varepsilon}(x) - \varepsilon, f_{n_\varepsilon}(x) + \varepsilon]$ from Theorem 3.1. we obtain

$$P\{f(x) \in I(n_\varepsilon)\} = P\left\{\left|f_{n_\varepsilon}(x) - f(x)\right| \le \varepsilon\right\} =$$

$$= \hat{Ð}\left\{ \grave{\sigma}^{-1} n_\varepsilon^{\frac{1-\beta}{2}} \left|f_{n_\varepsilon}(x) - f(x)\right| \le \frac{\sigma^{-1} n_\varepsilon^{\frac{1-\beta}{2}} \varepsilon}{a} a \right\} \ge P\left\{ \sigma^{-1} n_\varepsilon^{\frac{1-\beta}{2}} \left|f_{n_\varepsilon}(x) - f(x)\right| \le a \right\} \to 2 \quad (\ ) - 1 = \gamma$$

as $\varepsilon \to 0$. This sequential procedure has the disadvantage that n_ε depends on an unknown probability density $f(x)$ through σ^2, so instead n_ε consider the moment of stopping

$$N_\varepsilon = \inf\left(n \ge 1 : n \ge \left(\frac{a^2 k}{\varepsilon^2} f_n(x)\right)^{\frac{1}{1-\beta}} \right), \quad \varepsilon > 0. \qquad (7)$$

Then the sequential confidence interval for $f(x)$, based on the moment of stopping (7) has the form $I(N_\varepsilon) = [f_{N_\varepsilon}(x) - \varepsilon, f_{N_\varepsilon}(x) + \varepsilon],$ ε > 0 . Let's introduce the condition

(E): $\int_{-\infty}^{\infty} \left|K(t) - aK(at)\right| dt \le c_2 \dfrac{b}{1-a}, \int_{-\infty}^{\infty} |u|^s |\varphi(u)| \, du$ < ∞ для 0 < b < a < 1 и $s = 0,1,2,$ where c_2 is a positive constant.

Теорема 2.2. 1) If conditions (A) , (B), (C), (D), (E), (H) and $\dfrac{1}{2q+1} < \beta < \dfrac{1}{2}$ are met , then $f_n(x) \to f(x)$ with probability 1 for $n \to \infty$ and for some $\varepsilon_0 > 0$ $\sum_{m=1}^{\infty} \sup_{0 \le \varepsilon \le \varepsilon_0} P\left\{\dfrac{N_\varepsilon}{n_\varepsilon} \ge m\right\} < \infty$.

2) If conditions (A), (B), (C), (D) and $\frac{1}{2q+1} < \beta < 1$ are fulfilled, then $\zeta_\varepsilon(t), t \in [0,1] \xrightarrow{J} W(t), t \in [0,1]$ as $\varepsilon \to 0$,

where is the random process $\zeta_\varepsilon(\hat{E}) = \tilde{O}\sigma^{-1} t \cdot n_\varepsilon^{\frac{1-\beta}{2}} \cdot \sum_{i=1}^{[n_\varepsilon t]} Y_{n_\varepsilon}(\quad,\quad_i), \ \in [0,1]$ and $W(t), t \in [0,1]$ the Wiener process with $E(W(t)) = 0, E(W^2(t)) = 1.$

Hence, from Theorems 1.1., 1.2. and 1.3 we obtain following theorem.

Theorem 2.3. If conditions (A), (B), (C), (D), (E), (H) and $\frac{1}{2q+1} < \beta < \frac{1}{2}$ are fulfilled , then for n_ε from (6) and the stopping moment (7):

1) for any ε > 0 $P\{N_\varepsilon < \infty\} = 1$;

2) $N_\varepsilon \to \infty$ with a probability of 1 at $\varepsilon \to 0$;

3) $\lim\limits_{\varepsilon \to 0} E(N_\varepsilon) = \infty$;

4) $\frac{N_\varepsilon}{n_\varepsilon} \to 1$ with a probability of 1 at $\varepsilon \to 0$;

5) $\lim\limits_{\varepsilon \to 0} \frac{E(N_\varepsilon)}{n_\varepsilon} = 1.$

From Theorem 2 of the first paragraph, we obtain following theorem.

Theorem 2.4. Let conditions (A), (B), (C), (D), (H) and $\frac{1}{2q+1} < \beta < 1$ be fulfilled. Then

1) $\sigma^{-1} N_\varepsilon^{\frac{1-\beta}{2}} \left(f_{N_\varepsilon}(x) - f(x) \right) \xrightarrow{D} N(0,1)$ at $\varepsilon \to 0$;

2) $\lim\limits_{\varepsilon \to 0} P\left(f(x) \in I(N_\varepsilon) \right) \geq \gamma$, that is, the confidence interval of a fixed width $I(N_\varepsilon)$ is asymptotically consistent.

From Theorems 2.3. and 2.4. we obtain that if conditions (A), (B), (C), (D), (E), (H) and $\frac{1}{2q+1} < \beta < \frac{1}{2}$ are fulfilled, then the stopping moment (7) is asymptotically efficient in the sense of Chow and Robbins.

CONCLUSIONS AND SUGGESTIONS

The results of the study demonstrate that sequential fixed-width interval estimation provides a theoretically justified and practically efficient framework for constructing reliable confidence intervals in settings where classical fixed-sample methods are inadequate. By applying limit theorems for randomly stopped processes, the paper establishes general conditions under which sequential procedures achieve asymptotic consistency and efficiency for a wide class of statistical functionals. The application to kernel-based density estimation for linear processes further confirms that the sequential approach remains robust even in dependent data structures, offering a unified methodology for complex models.

To support further development of this field, it is essential to expand sequential estimation techniques to broader classes of dependent processes, explore adaptive bandwidth selection methods for nonparametric estimators, and integrate data-driven stopping rules capable of operating under minimal assumptions. Strengthening computational frameworks for real-time sequential inference and developing hybrid methods that combine classical and sequential techniques would significantly enhance the applicability of fixed-width interval estimation across modern statistical and econometric problems.

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